Reply to "Comment on 'Self-similarity and transport in the standard map'"

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In the Comment by Zumofen and Klafter (ZK) (Phys. Rev. E, preceding paper) the authors, comparing their results to those in a publication of ours [Benkadda, Kassibrakis, White, and Zaslavsky (BKWZ)] [Phys. Rev. E **55**, 4909 (1997)], state that "disagreement is regarded as resulting from conceptual differences in the approaches rather than from numerical inaccuracies." The papers discuss superdiffusion phenomena for the standard map. In fact, we will show here that the numerical results of ZK contradict neither the numerical results given in BKWZ nor the theory referenced there. At the same time we will indicate precisely why the theory used in ZK has restricted applicability and is not universal. We also provide additional analytical and numerical results in support of our conclusions. [S1063-651X(99)06803-8]

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Our reply is organized into sections on transport in phase space with a self-similar structure, oscillations in distribution data, comparison of the numerical results of Zumofen and Klafter (ZK) [1] with those of Benkadda, Kassibrakis, White, and Zaslavsky (BKWZ) [2], limitations of the theory used in [1], and some conclusions.

I. SELF-SIMILARITY

After numerous investigations of different area-preserving maps and equations, it became clear that particle transport can be non-Gaussian, i.e., anomalous (see [3,4], and references therein). A phenomenological approach to the problem was proposed, based on a fractional generalization of the Fokker-Planck-Kolmogorov [5] (FFPK) equation which we present for the one dimensional case in the form

$$\frac{\partial^{\beta_0} P(x,t)}{\partial t^{\beta_0}} = \mathcal{D}_{\alpha_0\beta_0} \frac{\partial^{\alpha_0} P(x,t)}{\partial |x|^{\alpha_0}} \quad (x,t\to\infty), \tag{1}$$

with fractional α_0,β_0, giving

$$\langle |x|^{\alpha_0}\rangle \sim \mathcal{D}_{\alpha_0\beta_0} t^{\beta_0} \tag{2}$$

or

$$\langle |x|^2\rangle \sim \mathcal{D}^{2/\alpha_0}_{\alpha_0\beta_0} t^{\mu}, \tag{3}$$

with transport exponent

$$\mu = 2\beta_0/\alpha_0 \tag{4}$$

if we accept the existence of the asymptotic self-similarity of $P(x,t)$. The coefficient

$$\mathcal{D}_{\alpha_0\beta_0} = \langle\langle |\Delta x|^{\alpha_0}/\Delta t^{\beta_0}\rangle\rangle \tag{5}$$

is defined through an elementary step $|\Delta x|$ performed during time interval Δt in the same sense as for the derivation of the usual FPK equation. There exists a very detailed discussion of limitations to the applicability of Eq. (1), and of how the limits $x,t\to\infty$ should be considered to obtain space-time self-similarity (see [3,4], and references therein). More importantly, the existence of different intermediate asymptotics in time was demonstrated for all kinds of distributions and their moments. This necessitates a careful use of the exponents μ, α_0,β_0 and any other significant parameters in attempts to understand the consequences of a proposed mechanism for the origin of the anomalous transport.

II. OSCILLATIONS

There were numerous observations that μ is not universal and depends on a control parameter K of the system. More precisely it was shown that μ depends on the phase space structure, on the vicinity of the system to some bifurcation values of K, etc. (see references in [3,4]). A crucial question is to find α_0,β_0 in Eqs. (1) and (2) from first principles, i.e., from dynamics, or more generally the set of α_0,β_j in

$$\langle |x|^{\alpha_0}\rangle = \text{const}\times \sum_j \mathcal{D}_{\beta_j} t^{\beta_j} \tag{6}$$

including the possibility of a spread of β near β_0, which actually is the case. (We simplify the situation and consider a fixed α_0 which also can be spread.)

Since the $\mu=\mu(K)$ is nonuniversal, we can select a *special value of K* for which a convenient approach can work. For the standard map we select $K=K_c=6.476\,939...$ for which there exists a hierarchical set of islands 5-11-11-11...

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in the phase space [2]. These islands belong to the accelerator mode, and their boundaries are sticky, giving rise to a superdiffusion with $\mu>1$. There are other special values of K in [3,4] which one can use, but the analytical approach should be specified depending on the situation.

In [6,7,3,4] a renormalization group (RG) approach was introduced for the case of the hierarchical island structure. The FFPK equation should be invariant under the $\hat{R}$ transform

$$\hat{R}: \quad \Delta x' = \lambda_S^{-1/2}\Delta x, \quad \Delta t' = \lambda_T \Delta t, \tag{7}$$

where $\lambda_S<1$ is a scaling parameter for the island area and $\lambda_T>1$ is the same for the period of an orbit near the island elliptic point. Both λ_S and λ_T can be obtained from the equations of motion or from a simulation with high accuracy. Applying $\hat{R}^n$ to Eq. (1) or Eq. (5) one can obtain a fixed point value in the limit $n\rightarrow\infty$,

$$\lim_{n\rightarrow\infty}(\lambda_S^{-\alpha_0/2}/\lambda_T^{\beta})^n = 1 \tag{8}$$

or for solutions of Eq. (8)

$$\beta_j = \frac{1}{2}\alpha_0\mu + 2\pi i j/\ln\lambda_T \quad (j=0,\pm1,\ldots), \tag{9}$$

with

$$\mu = 2\beta_0/\alpha_0 = |\ln\lambda_S|/\ln\lambda_T. \tag{10}$$

Applying Eq. (9) to Eq. (6), we obtain

$$\langle |x|^{\alpha_0}\rangle = \text{const}\times\mathcal{D}_{\beta_0}t^{\mu}\alpha_0/2 \times\left\{1+2\sum_{j=1}^{\infty}(\mathcal{D}_{\beta_j}/\mathcal{D}_{\beta_0})\cos\left(\frac{2\pi j}{\ln\lambda_T}\ln t\right)\right\}. \tag{11}$$

The expression obtained, in addition to the power law of transport with exponent μ, has a $\ln t$-periodic dependence with period

$$T_{\ln} = \ln\lambda_T. \tag{12}$$

The formula Eq. (11) provides an analytical expression for μ (see [3,4]) as well as log-periodic (in time) dependence of transport and all other probability distribution functions (PDF) including distributions of the Poincaré recurrences and exit times since all PDF satisfy the same RG transform Eq. (7) (see also [6,7]).

We also obtained the period $T_{\ln}$ which can be compared to the simulation results if α_0 is known. The formula Eq. (11) is a new one for the explanation of chaotic anomalous transport in dynamical systems, and its more detailed analysis will be considered elsewhere. Here, we have to mention only that the period $T_{\ln}$ in Eq. (11) is proportional to $\ln\lambda_T$. Just this property was derived in [3,4] and confirmed numerically to high accuracy for three periods both for the web map [3,4] and for the standard map [2]. The phenomenon of log periodicity is well known in the theory of phase transitions [8], structures with fractal geometry [9], branching random processes [10], Weierstrass random walk [11], dynamical systems [12–14], and other cases (see review [15]). The log-time periodicity was observed in our simulations [3,4,2,16] for chaotic dynamical systems and *it appears again in* [1] in their Figs. 2–4 (although without references). These simulations nontrivially confirm the existence of the RG property, and also raise the immediate question: *What slope of the oscillating curve should be compared to any given theory?* Even averaging over time is not trivial and depends on the time window.

III. DATA COMPARISON

Now we would like to discuss the numbers resulting from the simulations. Equation (10) for μ, obtained in [6,7], was applied to different cases when the renormalization theory for the island hierarchy can be applied. In particular, this comparison was made in [2] for the standard map with $K=K_c$. The results were $\mu=1.42\pm0.15$ (simulation) and 1.44 ± 0.02 [from Eq. (10)]. It is seen from Fig. 4 of [1] that $\mu\in(1.3–1.5)$ due to oscillations, which does not contradict our results (it is even mentioned in Fig. 4 of [1] that $\mu=1.4$).

Assuming the power tail distributions for the Poincaré-recurrence distribution $P_{\text{rec}}(t)$ and exit-time distribution $P_{\text{exit}}(t)$,

$$P_{\text{rec}}(t)\sim 1/t^{\gamma}, \quad P_{\text{exit}}(t)\sim 1/t^{\gamma_1}, \tag{13}$$

for the same case of hierarchical islands of the accelerator mode, it was shown in [3,4] that

$$\gamma = 2+\mu. \tag{14}$$

It was also explained in [3,4] that for a careful choice of a domain near an island boundary (in the trapping zone) one has $\gamma_1=\gamma$ and then $\gamma=\gamma_1=3.42\pm0.15$, which is in good agreement with the value $\gamma=3.5$ obtained in [2]. The reason for this comment was explained in great detail in [4,2]: *distributions of the Poincaré recurrences "collect" global information about the full phase space, while the exit-time distribution describes properties of a local domain from which exits are considered.* Nevertheless, the article [1] attacks the formula $\gamma_1=2+\mu$ [see Eq. (4) of [1]], neglecting our comments and without appropriate restrictions in application of the formula. That leads to an inferred discrepancy of some numerical results, which, in fact, *does not exist.*

Let us compare the simulation data of [1] (it is Fig. 2 of [1] which is relevant to us) to our results of [2]. The $\psi_{\text{stick}}(t)$ in [1] corresponds to our $P_{\text{exit}}(t)$ in Eq. (13). First of all, from Fig. 2 of [1], we have $\gamma_{\min}-1=1.2\pm0.1$ as the minimal value and $\gamma_{\max}-1=2.2\pm0.1$ as the maximal value. (The -1 occurs to convert between integral and differential probability distributions.) The large range $\gamma_{\min}$, $\gamma_{\max}$ is due to the oscillations. At this point we have to emphasize that all our estimations of γ were based on the separation of different periods, or hierarchical zones, and on considering separately $\gamma_{\min}$ (see Fig. 7 of [2]) and $\gamma_{\max}$ (see Fig. 8 of [2]). *The values $\gamma_{\max}$ and $\gamma_{\min}$ from Fig. 2 of* [1] *coincide with our results in Figs. 7 and 8 in* [2] within the error bars. The maximal slope (see Fig. 8 of [2]) corresponds to the period of time of the deepest penetration of flights into the hierar-

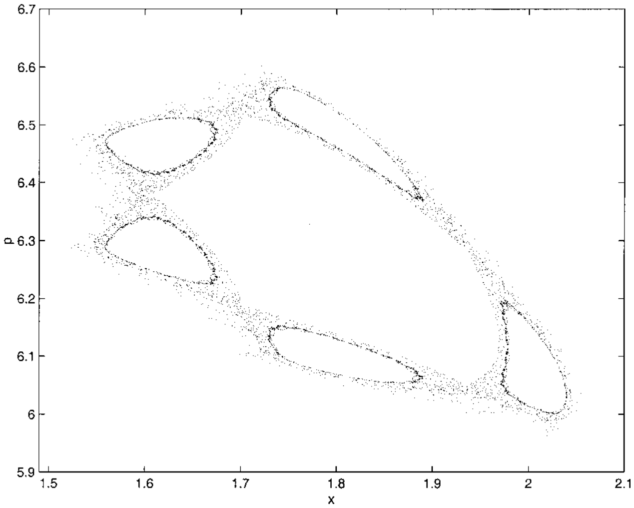

FIG. 1. A depiction of a flight trajectory of the length 1.2×10^4 steps (time) on the torus (x,p; mod 2π) displays a broad distribution in the phase space.

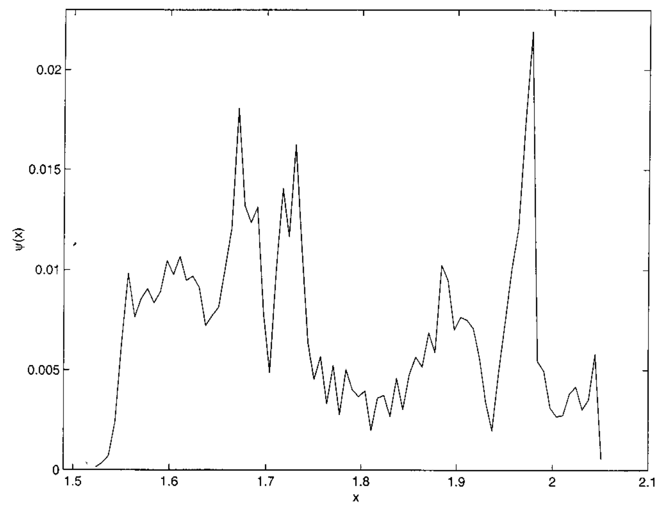

FIG. 2. Distribution of coordinates (80 bins) for the same flight as in Fig. 1 shows about 10% width relative to the full phase space and about 1% for peaks that correspond to penetration of the flight into the smaller island chain of the next generation (dark tiny islands in Fig. 1).

chical island structure. Note that only a specific range of time values can be discussed, the truly asymptotic value of the slope is not determined by the data, and asymptotic convergence arguments cannot bound the slope for any finite length experiment.

Concluding this part of the discussion, we arrive at approximately the same values for μ and γ if one considers the maximal values of the oscillating slopes of the distributions, as was remarked and emphasized in all our publications including the criticized article [2]. In other words, the difference in the results of simulations in [2] and [1], is negligible, contradicting the claim of [1] if one makes a correct comparison.

IV. LIMITATIONS TO *KZ* THEORY

Let us comment on the theory that has been used in [17,18] and [1]. We see two serious restrictions to the theory. The first one is related to Eqs. (8) and (9) of [1] where the trapping-time distribution

$$\psi(r,t) \sim t^{-\gamma_2} \delta(r - vt) \tag{15}$$

is proportional to a δ function with constant velocity v. This assumption is mentioned following Eq. (5) of [1]. In the standard map for the value $K_c = 6.476\,939$ corresponding to the accelerator mode the flights are in p and the ''velocity'' is an acceleration $a = K \sin x$, so in our case their distribution actually has the form

$$\psi(p,t) \sim t^{-\gamma_3} \delta(p - at), \tag{16}$$

with constant acceleration a.

However *use of distribution (16) or (15) for long time is not valid*. Natural limitations for the time of applicability can be obtained in different ways. Here, a straightforward simulation will be used. For $K = K_c$ we pick a trajectory with flights, cut a piece of the trajectory that corresponds to one flight of a length of 1.2×10^4 steps, and put this part of the trajectory on the torus $(x,p) \in (0,2\pi)$. The results are presented in Figs. 1–3. In Fig. 1 we see a fairly strong spread of the flight in the phase space with additional trapping in tiny next generation islands. In Figs. 2 and 3 we see distributions $\Psi(x)$, $\Psi(p)$ for the same flight. These distributions are very broad and the same breadth occurs in the distribution of the acceleration $a = \Delta p / \Delta t = K \sin x$. Observing other flights we conclude that there is always a dispersion Δa which for large observational time $t_0 \sim 10^6$ creates a huge dispersion in the argument of the δ function. This gives a limitation for the strong coupling model of [17,18] to

$$t_0 \lesssim 10^2 - 10^3. \tag{17}$$

The δ-function condition with constant v or constant a can be used only to describe processes in which the computational (or experimental) time is fairly short.

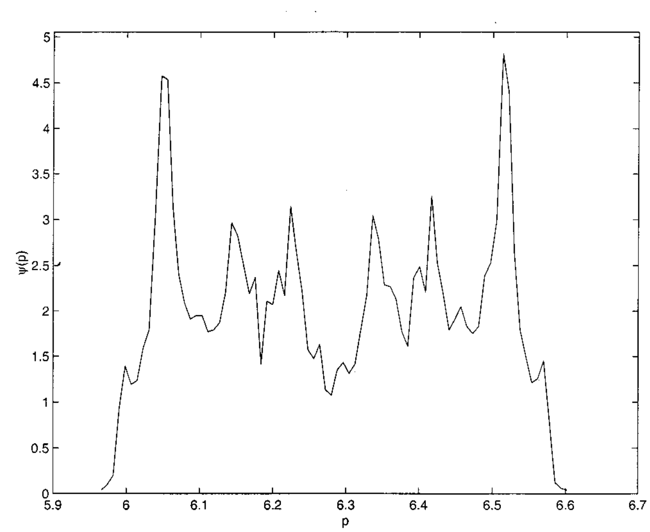

FIG. 3. The same as in Fig. 2 except for momentum. Ordinate is in values of 10^{-3}.

The second restriction comes from the way the result Eq. (11) of [1] was obtained in [17,18] (see also references in [1]). Starting from Eq. (7) of [1] and performing an inverse Fourier transform, the authors of [1] use an expansion of exp(*ikr*) in powers of *k*. Because of the presence of $\delta(r-vt)$ this means the occurrence of terms like $(kt)^n$. Then the limit $k\to 0$, $t\to\infty$ is not properly defined and there are different asymptotics. Their choice depends on a model and, particularly, on the velocity spread value Δv or the acceleration spread Δa in the case of the accelerator mode. The origin of this difficulty is due to the double limit $(r,t)\to\infty$ simultaneously but in different ways depending on the model.

We also need to mention that the result Eq. (11) of [1], even with restrictions, does not predict values of μ and γ but makes their connection within a fairly narrow interval due to the log-time oscillations.

V. CONCLUSION

In conclusion to our Reply, we would like to mention the following points.

(a) We have derived the log-time oscillations for the PDF and the moments, which were observed before in [2–4] and which also are displayed in Figs. 2–4 of [1]. The derivation assumes the existence of the RG transform Eq. (7). In this sense the curves, obtained in [1], confirm our previous results with more extensive computations.

(b) The values of μ and $\gamma_{\min}$, $\gamma_{\max}$ obtained from Fig. 2 of [1] reasonably coincide with those obtained in [2] if the same things are compared.

(c) Our simulation shows that the strong coupling model of [1] using the δ-function approximation for flights fails for a large time ($t>10^2-10^3$) due to the spreading of the velocity and acceleration of flights.

(d) The conflict of results appears in [1] when the theory of [17,18] is applied to a large time interval where the theory is not applicable. We assume that an additional time applicability restriction exists for the model of [1] arising from the conditions for the power expansion of the exp(*ikr*) used in [17,18]. We plan to consider this issue in more detail elsewhere.

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